Free wreath products as fundamental C^*-algebras

I / Compact quantum groups

<u>Def</u>: $C(G)$ unital C^*-alg $\Delta : C(G) \to C(G) \otimes C(G)$

→ coassociativity
→ cancellation

<u>Prop</u>: $\exists!$ $h : C(G) \to \mathbb{C}$ Haar state

$$(h \otimes id)\Delta = (id \otimes h)\Delta = h(\cdot)1$$

GNS → $C(G) \twoheadrightarrow C_r(G) \subseteq B(L^2G)$

$$L^\infty(G) = C_r(G)''$$

G is coamenable if $C(G) \twoheadrightarrow C_r(G)$ is an isom

G is hyperlinear if $L^\infty(G) \hookrightarrow \prod_\omega M$

Ex: $C(S_N^+)$

[Wang 98]

$$C(S_N^+) = \left\langle u_{ij} \quad 1 \le i,j \le N \;\middle|\; \begin{array}{l} u_{ij} = u_{ij}^* = u_{ij}^2 \\ \sum_i u_{ij} = \sum_i u_{ji} = 1 \end{array} \right\rangle$$

u a "magic unitary"

$$\Delta : u_{ij} \longmapsto \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

$$C(S_N^+) \xrightarrow{ab} C(S_N)$$ it is an is isomorphism iff $N \le 3$.

II / Graph of C^*-algebra:

Fima - Freslon - Germain

$\mathcal{G} = (V, E)$ connected $\quad \bar{e} \quad s(\bar{e}) = r(e)$

$r, s : E \to V$

$\forall p \in V \quad A_p$ a unital C^*-alg

$\forall e \in E \quad B_e$

$s_e, r_e : B_e \hookrightarrow A_p$.

Fundamental C^*-algebra of $\mathcal{G}$

→ $\mathcal{T}$ a maximal subtree

$$\pi_1(\mathcal{G}, \mathcal{T}) = \langle A_p ,\ p \in V ,\ u_e ,\ e \in E \rangle$$

full version

u_e unitaries $\quad u_e^* = u_{\bar{e}}$

if $e \in \mathcal{T}$, $\quad u_e = 1$

if $b \in B_e$, $\quad u_e^* s_e(b) u_e = r_e(b)$

→ ∃ reduced version

vN version

Ex: $\quad \mathcal{G} = A_1 \underset{B}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} A_2 \qquad \pi_1(\mathcal{G}) \cong A_1 \underset{B}{*} A_2$

$\vec{B}$ $\quad B \rightrightarrows A \qquad \pi_1(\mathcal{G}) \cong HNN(A, B, s, e)$

<u>Fact</u>: If every algebra A_p, B_e is CQG
s_e, r_e intertwine Δ's

$\to \exists! \ \Delta : \pi_1(\mathcal{G}) \longrightarrow \pi_1(\mathcal{G}) \otimes \pi_1(\mathcal{G})$
making it a CQG.

III / Free wreath <u>product</u>:

<u>Def</u>: Bichon 2004 $\qquad G$ a CQG $\quad N \geq 1$

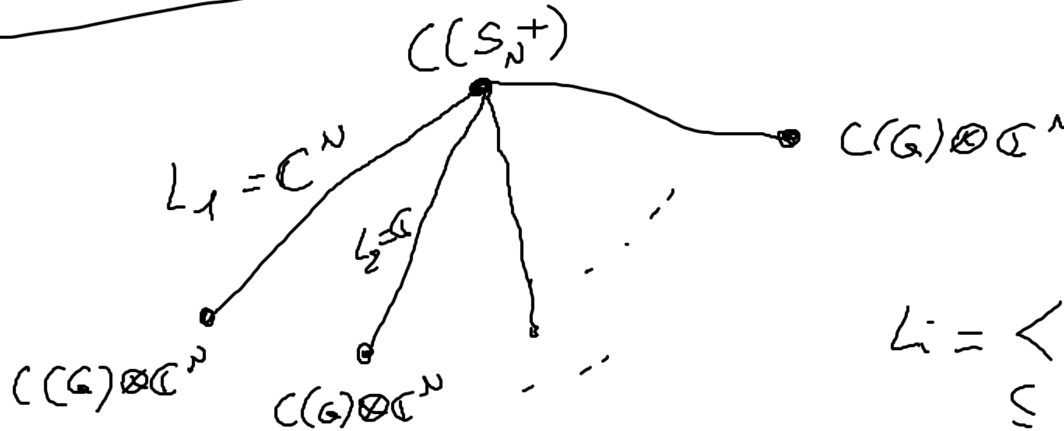
$$C(G \wr_* S_N^+) = C(G)^{*N} * C(S_N^+) \big/ I$$

$$I = \langle \nu_i(g) u_{ij} - u_{ij} \nu_i(g) \rangle_{1 \leq i,j \leq N,\ g \in C(G)}.$$

$\Delta_{|S_N^+} = \Delta_{S_N^+}$ $\nu_i : C(G) \longrightarrow C(G)^{*N}$

$$\Delta(\nu_i(g)) = \sum_{k=1}^{N} (\nu_i \otimes \nu_k)\Delta_G(g)(u_{ik} \otimes 1).$$

$\mathcal{G} =$



$L_i = \langle u_{ij},\ 1 \le j \le N \rangle \subseteq C(S_N^+)$

$\pi_1(\mathcal{G}) \cong C(G \wr_* S_N^+)$

$\pi_{1,red}(\mathcal{G}_{red}) \cong C_r(G \wr_* S_N^+)$

$\pi_{1,vN}(\mathcal{G}_{vN}) \cong L^\infty(G \wr_* S_N^+)$

Thm: (Fima-T)

The Haar state is the unique state $h \in C(G \wr_* S_N^+)^*$

s.t. $$h(\underbrace{a_1 \nu_{i_1}(b_1) a_2 \cdots a_{n-1} \nu_{i_n}(b_n) a_n}_{x}) = 0$$

$a_k \in C(S_N^+)$

$b_k \in C(G)$

as soon as x is reduced that is to say

$$h_G(b_k) = 0$$

and if $i_k = i_{k+1}$, then $\underbrace{E_{L_{i_k}}}_{C(S_N^+)}(a_k) = 0$

Prop: If $N=2$, $G \wr_* S_N^+ \cong G^{*2} \rtimes \mathbb{Z}_2$

Thm: • $L^\infty(G \wr_* S_N^+)$ has the Haagerup AP iff $L^\infty(G)$ has it.

• If G is Kac then $G \wr_* S_N^+$ is also Kac

$G \wr_* S_N^+$ is hyperlinear iff G is.

• K-amenable.

Results: - $C(S_N^+)$ has HAP $\forall N$ Brannan 12

- Hyperlinear Brannan-Chirvasitu-Freslon 20

- K-amenable Voigt

von Neumann algebras:

Assume $N \geq 4$, $C(G)$ is infinite dim. $M = L^\infty(G \wr_* S_N^+)$

Thm: M is a nonamenable, full, prime factor with no Cartan subalgebra.

- if G is Kac then it is II_1-factor
- if not then it is a III_λ factor $\lambda \neq 0$.

Known

↳ Factoriality for $N \geq 8$ → G a matrix CQG
↳ $\hat{\Gamma}$ Γ a discrete group

<u>Thm</u>: If $N \geq 2$

- If $L^{\infty}(G)$ is amenable and $C(G)$ infinite dim

  then the subalgebra $\langle \{ \partial_i(a) u_{ij} , 1 \leq j \leq N \} \rangle$

  is a maximal amenable subalg.

- if G is Kac then $L^{\infty}(S_4^{+}) \subseteq L^{\infty}(G) * S_4^{+})$

  is maximal amenable.

IV / K-theory :

$$
\begin{tikzcd}
\bigoplus_{i=1}^{N} K_0(\mathbb{C}^N) \arrow[r] & \bigoplus_{i=1}^{N} K_0(C(G)\otimes\mathbb{C}^N) \oplus K_0(C(S_N^+)) \arrow[r] & K_0(C(G \wr_* S_N^+)) \arrow[d] \\
K_1(C(G \wr_* S_N^+)) \arrow[u] & \bigoplus_{i=1}^{N} K_1(C(G)\otimes\mathbb{C}^N) \oplus K_1(C(S_N^+)) \arrow[l] & \bigoplus_{i=1}^{N} K_1(\mathbb{C}^N) \arrow[l]
\end{tikzcd}
$$

$H_N^{s+} = \widehat{\mathbb{Z}_s} \wr_* S_N^+ \leftarrow$ K-amenable

<u>Thm</u> :

$$
\begin{cases}
K_0(C_\bullet(H_N^{s+})) \simeq \mathbb{Z}^{sN^2-2N+2} \\
K_1(C_\bullet(H_N^{s+})) \simeq \mathbb{Z}
\end{cases}
\qquad 1 \leq s < +\infty, \quad N \geq 4
$$

Thm: $N, M \geq 8$, $s,t \geq 1$

$$C_r(H_M^{t+}) \cong C_r(H_N^{s+}) \quad \text{iff} \quad (M,t) = (N,s).$$

Rk: The same holds for $\widehat{\mathbb{F}_m} \wr_* S_N^+$.

If $G = \pi_1(\mathcal{G})$ $\quad$ $\mathcal{G}$ graph of CQG

If $C(H) \overset{\Delta's}{\hookrightarrow} C(G_1)$, $C(H) \hookrightarrow C(G_2)$ $\qquad G = \pi_1(\mathcal{G}) = G_1 \underset{H}{*} G_2$

$$G \wr_* S_N^+ \cong (G_1 \wr_* S_N^+) \underset{(H \wr_* S_N^+)}{*} (G_2 \wr_* S_N^+)$$

Prop: $G = \widehat{SL_2\mathbb{Z}} \wr_* S_N^+$ is K-amenable

$$\begin{cases} K_0(C_0(G)) \cong \mathbb{Z}^{8N^2-2N+2} \\ K_1(C_0(G)) \cong \mathbb{Z} \end{cases}$$